

Relationship between spin squeezing and single-particle coherence in two-component Bose-Einstein condensates with Josephson coupling

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We investigate spin squeezing of a two-mode boson system with a Josephson coupling. An exact relation between the squeezing and the single-particle coherence at the maximal-squeezing time is discovered, which provides a more direct way to measure the squeezing by readout the coherence in atomic interference experiments. We prove explicitly that the strongest squeezing is along the J_z axis, indicating the appearance of atom number-squeezed state. Power laws of the strongest squeezing and the optimal coupling with particle number N are obtained based upon a wide range of numerical simulations.

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I. INTRODUCTION

Spin squeezing is a nonclassical effect of collective spin systems [1, 2, 3, 4], showing reduced spin fluctuation in one certain spin component normal to the mean spin. Kitagawa and Ueda proposed spin squeezing generated by the self-interaction Hamiltonian $H_1 = 2\kappa J_z^2$, due to the so-called one-axis twisting (OAT) effect [1]. The OAT-type spin squeezing could be realized in weakly interacting Bose-Einstein Condensate (BEC) [5], or atomic ensemble in a dispersive regime [6]. The self-interaction H_1 also leads to phase diffusion of the BEC [7], which indicates a decay of single-particle coherence [8, 9, 10, 11, 12, 13].

Beyond the OAT model, an Josephson-like coupling (JLC) term ΩJ_x was added to the Hamiltonian H_1 with purpose to coherently control the phase diffusion [13] and the spin squeezing [14, 15, 16]. It was shown that the JLC model [see Eq.(1)] results in strong reduction of spin fluctuation along the z (i.e., J_z) axis, provided that the additional field is tuned optimally [15]. We found the maximal-squeezing time t_0 of the JLC model, and proposed a simple scheme to store the strongest squeezing along the z axis for a long time [16]. So far, there remain certain questions unsolved: Is there any relation between the squeezing and the single-particle coherence? In addition, to what degree can the strongest squeezing reach in the JLC model? The first question is important because it relates to measurement of the squeezing.

In this paper, we present an exact relation between the squeezing and the coherence by solving the Heisenberg equation. Our results show that local minima of the squeezing and the coherence occur simultaneously for the coupling Ω larger than its optimal value Ω_0 . Unlike the OAT scheme, where number variance ΔJ_z is time-independent, we prove explicitly that the squeezing at time t_0 is along the z axis in the JLC model [17]. The strongest squeezing obeys the power law $\xi_0 = \Delta J_z(t_0)/\sqrt{j/2} \propto N^{-1/3}$, which can be measured by readout the single-particle coherence through the vis-

ibility of the interference fringe [8, 9, 10, 11, 12].

Our paper is organized as follows. In Sec. II, we introduce theoretical model and derive some formulas for the single-particle coherence and the squeezing parameter. In Sec. III, quantum dynamics of the coherence and the squeezing are investigated for the OAT and the JLC models, respectively. In Sec. IV, we present exact relation between the coherence and the spin squeezing at the maximal-squeezing time t_0 . In Sec. V, power rules of the optimal coupling and the strongest squeezing as a function of particle number N are investigated based upon a wide range of numerical simulations. Moreover, we compare numerical result of t_0 for the optimal coupling case with its analytic solution. Finally, a summary of our paper is presented.

II. THEORETICAL MODEL AND SOME FORMULAS

To begin with, we consider a two-component weakly-interacting BEC consisting of $2j$ atoms in two hyperfine states $|1\rangle$ and $|2\rangle$ coupled by a radio-frequency (or microwave) field [18, 19]. A tightly-confined BEC can be described by the JLC Hamiltonian ($\hbar = 1$) [20]

$$H_2 = \Omega J_x + 2\kappa J_z^2, \quad (1)$$

where the angular momentum operators $J_+ = (J_-)^\dagger = \hat{a}_2^\dagger \hat{a}_1$, $J_z = (\hat{a}_2^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_1)/2$ obey the SU(2) Lie algebra. The total particle number $N = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2$ is a conserved quantity. In Eq.(1), we have neglected the term proportional to J_z by assuming equal intraspecies atom-atom interaction strengths [5]. The Rabi frequency Ω can be controlled by the strength of the external field. The self-interaction term $2\kappa J_z^2$ leads to spin squeezing, which is quantified by a parameter [1]:

$$\xi = \frac{\sqrt{2}(\Delta J_n)_{\min}}{j^{1/2}}, \quad (2)$$

where $j = N/2$ and $(\Delta J_{\mathbf{n}})_{\min}$ represents the minimal variance of a spin component $J_{\mathbf{n}} = \vec{J} \cdot \mathbf{n}$ normal to the mean spin $\langle \vec{J} \rangle$. The coherent spin state (CSS), defined formally as [21]

$$|\theta, \phi\rangle = e^{-i\theta(J_x \sin \phi - J_y \cos \phi)} |j, -j\rangle \quad (3)$$

has the minimal variance $(\Delta J_{\mathbf{n}})_{\min} = \sqrt{j/2}$ and $\xi = 1$. Therefore, a state is called spin squeezed state if its variance is smaller than that of the CSS, i.e. $\xi < 1$. Besides the squeezing, the self-interaction $2\kappa J_z^2$ also leads to the phase diffusion, which indicates a decay of the single-particle coherence. Such a kind of coherence is measured by off-diagonal elements of the single-particle density matrix $\rho_{ij}^{(1)} = \langle a_i^\dagger a_j \rangle / N$ with $i, j = 1, 2$. Formally, one introduces the first-order temporal correlation function [13]:

$$g_{12}^{(1)} = \frac{|\rho_{12}^{(1)}|}{\sqrt{\rho_{11}^{(1)} \rho_{22}^{(1)}}} \equiv \frac{|\langle J_+ \rangle|}{\sqrt{j^2 - \langle J_z \rangle^2}}, \quad (4)$$

which is observable in experiments by extracting the visibility of the Ramsey fringes [8, 9, 10, 11, 12]. One of the goals of this paper is thereby to present the relation between the squeezing ξ and the first-order coherence $g_{12}^{(1)}$.

Let us first examine the exact numerical solutions of the time-dependent Schrödinger equation governed by the JLC Hamiltonian H_2 . We consider that the spin system starts from the lowest eigenvector of J_x , $|j, -j\rangle_x = e^{-i\pi J_y/2} |j, -j\rangle$, a particular CSS, Eq. (3), with $\theta = \pi/2$ and $\phi = \pi$. Such an experimentally realizable state can be prepared by applying a two-photon $\pi/2$ pulse to the ground state $|j, -j\rangle$ with all the atoms in the internal state $|1\rangle$ [5, 12, 18]. The spin state at arbitrary time t can be expanded as: $|\Psi\rangle = \sum_m c_m |j, m\rangle$, and the amplitudes c_m obey $i\dot{c}_m = \varepsilon_m c_m + X_{-m} c_{m+1} + X_m c_{m-1}$, where $\varepsilon_m = 2\kappa m^2$, $X_m = \frac{\Omega}{2} \sqrt{(j+m)(j-m+1)}$ with $X_{-j} = 0$. The amplitudes of the initial CSS are $c_m(0) = \langle j, m | j, -j \rangle_x = \frac{(-1)^{j+m}}{2^j} \binom{2j}{j+m}^{1/2}$. Obviously, $c_{-m}(0) = c_m(0)$ for even N and $c_{-m}(0) = -c_m(0)$ for odd N , which gives the expectation value $\langle J_z(0) \rangle = 0$ and the variance $\langle J_z^2(0) \rangle = j/2$. Note that some references adopt the Hamiltonian $H_3 = -\Omega J_x + 2\kappa J_z^2$ to investigate the BEC in a double-well potential [20], which corresponds to H_2 for $\Omega < 0$ case. In this work, we consider only positive Ω and κ by assuming repulsive atom-atom interactions. If either Ω or κ is negative, our results remain valid by using initial state $|j, j\rangle_x$ [22].

Since $c_{-m}(0) = \pm c_m(0)$ and $X_{\pm m} = X_{\mp m+1}$, we introduce linear combinations of the amplitudes $p_m^{(\pm)} = c_m \pm c_{-m}$. For even N case, one can derive a closed set of equations for $p_m^{(-)}$. However, all $p_m^{(-)}(0) = 0$ lead to $p_m^{(-)}(t) = 0$, thus $c_{-m}(t) = c_m(t)$. As a result, dynamical evolution of the even N system is determined solely by the equations of the amplitudes $p_m^{(+)}$ with $m = 0, 1, \dots, j$. Similarly, for the odd N case, we obtain

$p_m^{(+)}(0) = p_m^{(+)}(t) = 0$, i.e., $c_{-m}(t) = -c_m(t)$. Quantum dynamics of the odd N system depends on the equations of $p_m^{(-)}$. The above processes have certain advantages to: (i) reduce the total Hilbert space dimension from $2j+1$ to $j+1$ (even N) or $j+1/2$ (odd N); (ii) Since $c_{-m} = \pm c_m$, we obtain $\langle J_y \rangle = \langle J_z \rangle = 0$ and $\langle J_x \rangle = \langle J_+ \rangle \neq 0$, i.e., the mean spin is always along the x axis. Actually $\langle J_+ \rangle$ is a real function; (iii) The correlation function is simplified as $g_{12}^{(1)} = j^{-1} |\langle J_x \rangle|$, and the spin component normal to the mean spin is $J_{\mathbf{n}} = J \cdot \mathbf{n} = J_y \sin \theta - J_z \cos \theta$. The variance of $J_{\mathbf{n}}$ is $(\Delta J_{\mathbf{n}})^2 = \langle J_{\mathbf{n}}^2 \rangle - \langle J_{\mathbf{n}} \rangle^2 = (C - A \cos 2\theta - B \sin 2\theta)/2$, where $A = \langle J_y^2 - J_z^2 \rangle$, $B = \langle J_z J_y + J_y J_z \rangle$, and $C = \langle J_y^2 + J_z^2 \rangle$. From the relation $\partial_\theta (\Delta J_{\mathbf{n}})^2|_{\theta_{\min}} \equiv 0$, we get $\tan(2\theta_{\min}) = B/A$ and the minimal variance $(\Delta J_{\mathbf{n}})_{\min}^2 = (C - \sqrt{A^2 + B^2})/2$.

III. QUANTUM DYNAMICS OF THE COHERENCE AND THE SQUEEZING

The OAT model H_1 (i.e., $\Omega = 0$) can be solved exactly in Heisenberg picture [1], with its analytic results: $A = (j/2)(j - 1/2) [1 - \cos^{2j-2}(4\kappa t)]$, $B = -j(2j - 1) \sin(2\kappa t) \cos^{2j-2}(2\kappa t)$, and $C = j + A$ due to the time-independent variance $\langle J_z^2 \rangle = j/2$. The strongest (optimal) squeezing $\xi_0 = \xi(t_s) \simeq (4/3)^{1/6} N^{-1/3}$ occurs at time $t_s \simeq 6^{1/6} N^{-2/3}/2$. The single-particle coherence $g_{12}^{(1)}(t) = \cos^{N-1}(2\kappa t) \simeq e^{-(t/t_d)^2}$ with the phase-diffusion time $\kappa t_d = (2N)^{-1/2}$. Obviously, $g_{12}^{(1)}(t_d) = e^{-1} g_{12}^{(1)}(0) = 1/e$. The coherence $g_{12}^{(1)}(t)$ has been measured in experiment by extracting the visibility of the Ramsey fringe [12]. As shown in Fig. 1(a), the optimal squeezing occurs within the coherence time due to $t_s < t_d$. Moreover, the coherence $g_{12}^{(1)}$ decays to zero at $t_0 = \pi/(4\kappa)$, and recover to unity at $2t_0$ [not shown in Fig. 1(a), see Refs.[7, 13]].

Except for $N = 2, 3$ cases, the JLC model ($\Omega \neq 0$) can not be solved exactly [15, 16, 23]. Numerical simulations of the single-particle coherence $g_{12}^{(1)}(t)$ and the squeezing $\xi(t)$ are presented in Fig. 1(b)-(d) for $N = 40$ and various Ω . Similar with the OAT case, the coherence $g_{12}^{(1)}(t)$ collapses to its local minimum at t_0 then revives partially at about $2t_0$, and the maximal squeezing occurs at t_s for a small coupling $\Omega = 2\kappa$ [Fig. 1(b)]. Two time scales t_0 and t_s tend to merge with the increase of Ω . For $\Omega \geq \Omega_0$, both the coherence $g_{12}^{(1)}$ and the squeezing ξ reach local minima at the same time t_0 [Fig. 1(c) and (d)]. Here Ω_0 is the optimal coupling to produce the strongest squeezing in the JLC model, such as $\Omega_0 = 4.2405\kappa$ for $N = 40$.

It should be mentioned that the spin state at t_0 exhibits a very sharp probability distribution and a strong reduction of the number variance ΔJ_z [16]. The loss of the coherence (or visibility) as an evidence of the number squeezing at time t_0 , as shown in Fig. 1(b)-(d), has been observed by Orzel et al. [8]. Moreover, our results show that the phase diffusion is suppressed due to the appear-

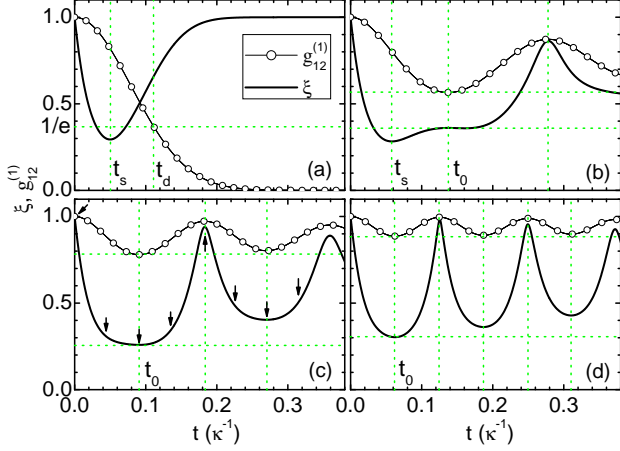


FIG. 1: Time evolution of the squeezing parameter ξ (solid thick lines) and the single-particle coherence $g_{12}^{(1)}(t)$ (thin lines with open circles) for $N = 40$ and various Rabi frequencies: (a) $\Omega = 0$, (b) $\Omega = 2$, (c) $\Omega = \Omega_0 = 4.2405$ (optimal coupling), (d) $\Omega = 8$. Time t is in units of κ^{-1} , and the units of Ω is κ .

ance of number-squeezed state at t_0 , consistent with the experimental observations [11]. In experiments so far, however, the number squeezing were detected through the observation of increased phase fluctuation $\Delta\phi$, or through an increased phase-diffusion time [24]. In fact, the number fluctuation has a nontrivial relation with the phase fluctuation, complex even for a single-mode light field [25]. Consequently, a more direct way to measure the number fluctuation ΔJ_z is necessary.

IV. EXACT RELATION BETWEEN THE COHERENCE AND THE SQUEEZING AT THE MAXIMAL-SQUEEZING TIME

There exists an exact relation between the coherence $g_{12}^{(1)}$ and the squeezing ξ at time t_0 . To see this, let us examine the Heisenberg equations: $\dot{J}_x = -2\kappa(J_z J_y + J_y J_z)$ and $\dot{J}_z = \Omega J_y$. The first equation gives the relation between the coherence $g_{12}^{(1)}$ and the squeezing angle θ_{\min}

$$\frac{d}{dt}g_{12}^{(1)} = 2\kappa A j^{-1} \tan(2\theta_{\min}). \quad (5)$$

As $g_{12}^{(1)}$ reaches its local minimum at t_0 , $(dg_{12}^{(1)}/dt)_{t_0} \equiv 0$, then the squeezing angle $\theta_{\min} = 0$ (or π) provided $A \neq 0$. Combining the two Heisenberg equations, we obtain further $dJ_z^2/dt = \Omega(J_z J_y + J_y J_z) = -(\Omega/2\kappa)dJ_x/dt$, which yields $\langle J_z^2 \rangle = \lambda - \Omega \langle J_x \rangle / (2\kappa)$ with the integral constant λ . For the initial CSS $|j, -j\rangle_x$, we have $\lambda = j(1 - \Omega/\kappa)/2$. At time t_0 , $B = 0$ and $A > 0$, and the minimal variance $(\Delta J_n)_{\min}^2 = (C - A)/2 \equiv \langle J_z^2 \rangle$, i.e., the squeezing along the z axis [17]. As a result, we obtain a simple relation

between $g_{12}^{(1)}$ and ξ

$$\xi^2(t_0) = \frac{2\langle J_z^2(t_0) \rangle}{j} = 1 - \frac{\Omega}{\kappa} \left[1 - g_{12}^{(1)}(t_0) \right], \quad (6)$$

which is valid for arbitrary Ω . For instance $\Omega = 0$, $g_{12}^{(1)} = 0$ and $\xi = 1$ at $t_0 = \pi/(4\kappa)$; while for $\Omega > N\kappa$, from Eq. (6) we get $g_{12}^{(1)}(t_0) \simeq \xi(t_0) \simeq 1$; two trivial results due to weak squeezing. Hereafter, we focus on the coupling around its optimal value Ω_0 , with which the strongest squeezing $\xi_0 = \xi(t_0, \Omega_0)$ can be obtained at time t_0 . According to Eq. (6), one can measure ξ_0 by readout the coherence $g_{12}^{(1)}$ in atomic interference experiments [8, 9, 10, 11, 12].

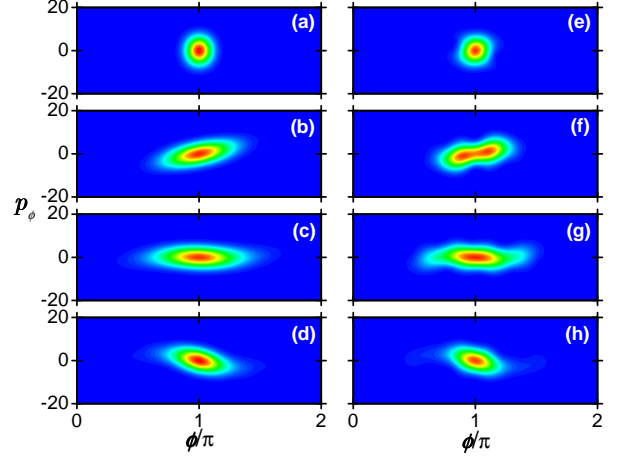


FIG. 2: (color online) Husimi Q function in the phase space (ϕ, p_ϕ) for various times: (a) $t = 0$, (b) $t = 0.045$, (c) $t = t_0 = 0.09064$ (the optimal squeezing), (d) $t = 0.135$, (e) $t = 0.183127$, (f) $t = 0.225$, (g) $t = 0.26988$, and (h) $t = 0.315$, corresponding to the times indicated by the arrows in Fig. 1(c), respectively. Other parameters are $N = 40$, $\Omega = \Omega_0 = 4.2405\kappa$, and time t is in units of κ^{-1} .

The time scale t_0 can be obtained based upon the phase model [16]. By replacing $J_z \rightarrow p_\phi = -i\partial_\phi$ and $J_x \rightarrow j \cos \phi$, the JLC Hamiltonian H_2 can be rewritten as $H_\phi = 2\kappa p_\phi^2 + \Omega j \cos \phi$ [26], where ϕ is relative phase between two bosonic modes. The phase model allows us to treat the spin system as a simple classical pendulum oscillating around the minimum of the Mathieu potential $\cos \phi$, i.e., $\phi = \pi$. In the Josephson regime $1 < \Omega/\kappa \ll N$, the pendulum rotates in the phase space (ϕ, p_ϕ) with the effective frequency $\omega_{\text{eff}} = \sqrt{2\kappa\Omega N}$ [16, 27]. To illustrate this motion, we calculate the Husimi Q function

$$Q(\theta, \phi) = |\langle \theta, \phi | \Psi(t) \rangle|^2 \quad (7)$$

in the phase space (ϕ, p_ϕ) , where $p_\phi = -j \cos(\theta)$ describes the population imbalance of the two modes, and the polar angle ϕ represents the relative phase [28]. The CSS $|\theta, \phi\rangle$ is given in Eq. (3). As shown in Fig. 2(a), the Q function is a circle for the initial CSS, which represents Poisson distribution of the number variance $(\Delta J_z)^2$ and

the phase uncertainty $(\Delta\phi)^2$ [12]. As time increases, it becomes an elliptic shape [Fig. 2(b)], rotating clockwise in the phase space. After a duration t_0 , the ellipse elongates horizontally corresponding to the optimal squeezing of $(\Delta J_z)^2$ [Fig. 2(c)]. It seems reasonable to suppose that the motion of the ellipse is consistent with that of the pendulum. In fact, the trajectory of the pendulum is just passing through the horizontal axis ($p_\phi = 0$) at time $T/4$, where $T = 2\pi/\omega_{\text{eff}}$ is the period of the pendulum. As a result, we get the maximal-squeezing time [16]

$$\kappa t_0 \simeq \kappa \frac{T}{4} = \frac{\pi}{2} \sqrt{\frac{\kappa}{2\Omega N}}, \quad (8)$$

which is valid for large N ($\geq 10^3$). As shown in Fig. 2(e)-(h), for $t \geq 2t_0$ ($\simeq T/2$) the Q functions almost recover to original shapes, due to partial revival of the squeezing ξ and the coherence $g_{12}^{(1)}$ [see Fig. 1(c)].

V. OPTIMAL COUPLING AND THE STRONGEST SQUEEZING

To create the strongest reduction of $(\Delta J_z)^2$ as shown in Fig. 2(c), we need to determine the optimal coupling Ω_0 as a function of particle number N . Note that the optimal squeezing occurs at t_s for the OAT and t_0 for the JLC, respectively. For large N , the latter time scale should be comparable with the former one as $\Omega = \Omega_0$. Such a non-rigorous comparison enables us to suppose a power law as $\Omega_0/\kappa \simeq N^{1/3}$. Numerical solution of Ω_0 is presented in Fig. 3 for N up to 2×10^5 . We fit the data as $\Omega_0/\kappa = aN^b$ and find the power-exponent $b = 0.32655$, very close to the expected value $1/3$. From the inset of Fig. 3, we also find that the larger number N is adopted, the better fit is obtained.

In Fig. 4, we investigate the optimal squeezing ξ_0 as a function of N . The fitting result is $\xi_0 \simeq 0.8578N^{-1/3}$, which is slightly smaller than the OAT result $\xi_0 \simeq (4/3)^{1/6}N^{-1/3} \simeq 1.0491N^{-1/3}$. Small difference of ξ_0 between the OAT and the JLC does not deteriorate the advantages of the latter scheme. In fact, there is no number squeezing in the OAT model due to $[J_z^2, H_1] = 0$. In our case, one can realize $\langle J_z^2(t_0) \rangle = \xi_0^2 \langle J_z^2(0) \rangle$ with $\xi_0 < 1$, indicating the appearance of the number-squeezed state [17]. Such a kind of squeezed state has been observed in optical lattices [8], optical trap [10], and atom chip [11]. However, the observed squeezing $\xi_0 \simeq 0.1$ for $N = 4 \times 10^5$ [11], weaker than our result $\xi_0 \simeq 1.467 \times 10^{-2}$ for $\Omega_0 = 58.05\kappa$ and $N = 2 \times 10^5$. Finally, within inset of Fig. 4, we show the time scale t_0 as a function of N . Inserting $\Omega_0/\kappa = N^{1/3}$ into Eq. (8), we find that analytic expression of t_0 gives good agreement with the exact numerical simulations.

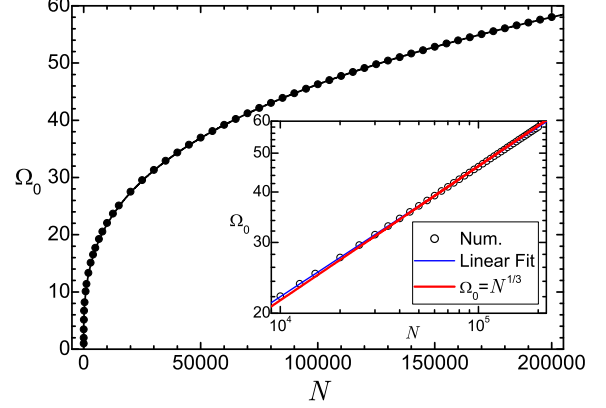


FIG. 3: (color online) Numerical simulation of the optimal coupling Ω_0 as a function of N (solid circles) in the normal scale. The solid line is the fit to power law $\Omega_0 = aN^b$ with $a = 1.07827$ and $b = 0.32655$. The inset: comparison linear fit (blue line) of numerical results (open circles) with $\Omega_0 = N^{1/3}$ (red line) in the log-log scale. Ω_0 is in units of κ .

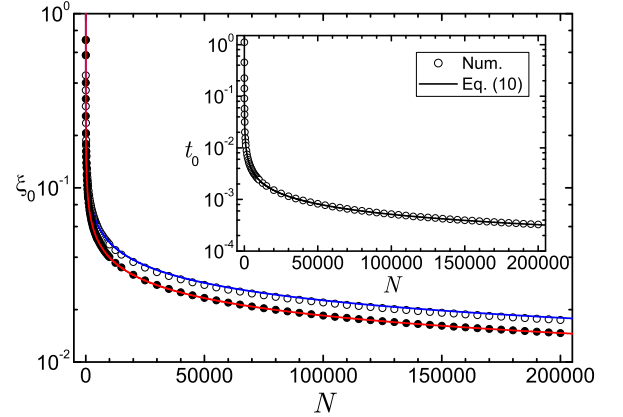


FIG. 4: (color online) The optimal squeezing ξ_0 for the OAT (open circles) and the JLC (solid circles) as a function of N . The blue line is given by $\xi_0 \simeq (4/3)^{1/6}N^{-1/3} \simeq 1.0491N^{-1/3}$; the red line is a fitting curve as $\xi_0 = 0.8578N^{-1/3}$. The inset: numerical simulations of t_0 (open circles), and the analytic result of Eq. (8) with $\Omega_0/\kappa = N^{1/3}$ (solid line) as a function of N for optimal squeezing. The time t_0 is in units of κ^{-1} .

VI. CONCLUSION

In summary, we have investigated optimal spin squeezing in a two-component Bose-Einstein condensate with a Josephson coupling. We show that: (i) the squeezing ξ at time t_0 aligns along the z axis, which is equivalent with the number squeezing [8, 9, 10, 11, 17], and is desirable for high-precision atom interferometry [28, 29]; (ii) there exists a simple relation between the squeezing ξ and the

single-particle coherence $g_{12}^{(1)}$ at t_0 , Eq.(5) and Eq.(6), from which it is possible to measure the number variance ΔJ_z by readout the coherence $g_{12}^{(1)}$ in the interference experiments; (iii) the strongest squeezing with the power law $\xi_0 \simeq 0.8578N^{-1/3}$ is achievable by applying the optimal coupling $\Omega_0/\kappa \simeq N^{1/3}$. We also discuss the maximal-squeezing time t_0 via the phase model and the Husimi Q function, and find that analytic result, Eq. (8), agrees with its numerical simulations.

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